Integral Equations Revisited

INTRODUCTION

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Technical Article

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Introduction

It is customary to express a field problem mathematically as either a differential equation or as an integral equation. For example, if a static field solution is required, then either Laplace's differential equation or an integral equation based on Green's theorem are used, i.e.

$$\nabla^2 \phi = 0 \qquad (1)$$

or

$$\phi = \frac{1}{4\pi} \int_{\Gamma} \left(\frac{1}{R} \frac{\partial \phi}{\partial n} - \frac{\partial}{\partial n}\left(\frac{1}{R}\right)\phi \right) d\Gamma \qquad (2)$$

where the symbols have their usual meaning.

The essential difference between these two formulations stems from the fact that the differential equation describes the local and the integral equation the global behaviour respectively. Accordingly Laplace equation, in itself, is not sufficient to solve the problem as boundary conditions have to be specified additionally. Whereas the integral equation alternative describes the physics of the problem in a compact form including the boundary conditions. Thus the condition that the potential is regular at infinity (ϕ=0) is implied by the equation but for a unique solution either the potential or its normal derivative must be specified, within the equation, for all physical surfaces.

The effect of the above on the numerical solution procedures is profound. As a consequence of the local character inherent in the differential case the whole of space, including free space, within the bounding surfaces needs to be discretised and it must be remembered that the bounding surface may reach out to infinity. On the other hand, the global character of integral equations confines the discretisation to the active regions, i.e. sources, conductors and permeable material, therefore free space is not discretised. Furthermore, if the problem is homogeneous and linear, as is the case in equations (1) & (2), then the domain of integration is further reduced from a volume to a surface in 3d and from an area to a line in 2d.

Integral versus Differential

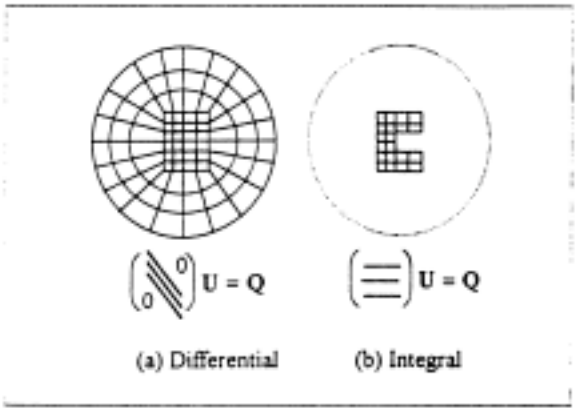


Fig. 1 Unbounded problem discretized into elements

The choice of method has an important effect on the matrix structure and size. In the differential case both the free space and active regions are modelled leading to a larger but sparsely populated matrix. This is in contrast to the integral case where only the active regions are modelled but now the matrix, though smaller, is fully populated. This reflects the nature of the two formulations whereby fields are related locally in the first case but globally in the second. The effect on discretisation is shown in Figure 1.

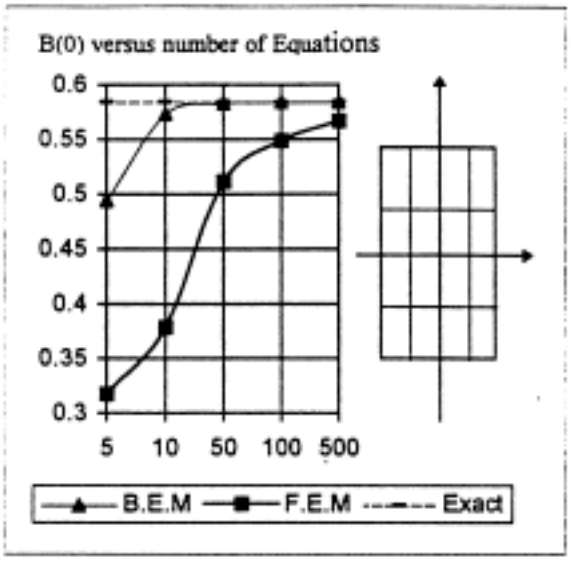


Fig. 2 Comparison of differential & integral solutions

The structure and size of the matrix will effect the number of numerical operations required for the solution and the computer memory storage requirements respectively. The progress toward convergence of the two methods is directly compared in Figure 2 using a rather trivial example of a rectangular ferromagnetic conductor transporting a constant current. At first sight this result appears to favour the integral approach with its fast smooth convergence to the correct solution for a modest discretisation. However, it must be born in mind that the fully populated solution in the above example was achieved using the standard algorithm for solving dense matrices, Gaussian elimination, which has an operation count n^3 whereas for the sparse matrix an iterative process which depends upon $N \log N$ was used [1]. Thus, on these grounds only, extrapolating the problem size indefinitely it can be surmised that the differential method will ultimately win. At what level of discretisation this occurs depends upon many factors as in general $n<<N$. A further point relates to whether the computer is serial or parallel and indeed whether Gaussian elimination really is the best algorithm if the matrix has an exploitable structure. This point will be returned to later.

Apart from the operation count and the properties of the system matrix there are several other considerations t:

discuss when comparing the two approaches. The main properties are listed in Table 1. Early implementations of integral methods tended to use a point matching technique in which the equation was simply matched to the discretised points thus avoiding the multiple integrations of singular kernels but in differential methods the integrations of the polynomial basis functions and test functions could be done with ease.

	Differential	Integral
Matrix	Sparse & banded	Full
Basis & test functions	Normally polynomial	Limited in practice
Matrix coefficients	Elementary	Multiple integrations
Integrand	Polynomial	f(1/r)
Source	N	n
Set-up	$O(N^2)$	$O(n^2)$
Solution	O(N log N)	$O(n^3)$
Discretisation	All space	Active regions
Far-field	Bounded	Unbounded
Field recovery	Readily available	Post-processing
Non-linearity	Easy	Troublesome
Smoothness	Piecewise	Continuous
Motion	Meshing problems	No meshing problems
Beam tracking	Field gradients difficult	Field gradients easy
Optimisation	Meshing problems	No meshing problems
Parallelisation	Complicated	Intrinsic

Table 1. Comparison specifics

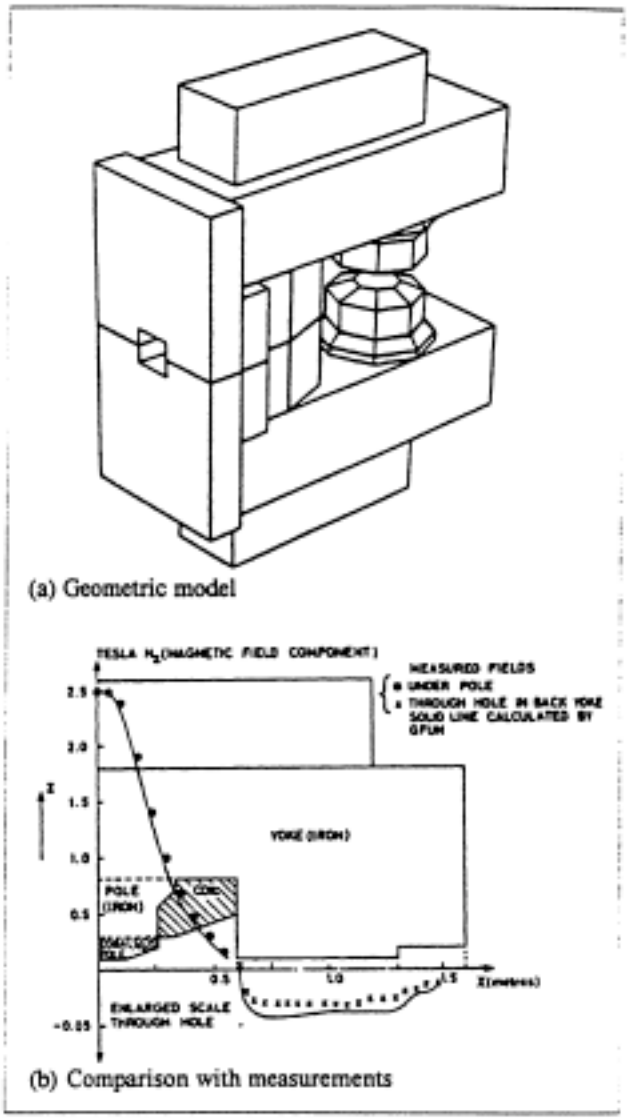
Historical Perspective - Magnetostatics

Since the early 1970's software based on the differential equations (mostly using the finite element method) has been progressively introduced to industry with striking success. The early achievements with two dimensional problems consolidated this approach and today whilst three dimensional solutions are almost routine the ability of software in generating complex 3d models remains a limiting issue. In fact this issue was faced by the early developers in the decision process of how to extend the finite element method to be able to handle 3d problems [2]. One example was in the area of magnet design for the large scale particle accelerators where accurate 3d field predictions were needed in order to minimise the building of costly prototypes. The extension of the very successful 2d codes to 3d in those days was a daunting task because, apart from lack of computing power, the problem of mesh generation was beyond the technology available. This was because meshes were required in all space - a basic consequence of the differential approach. This is why the first magnetostatic code in 3d to be routinely used for this class of problem was based on the magnetisation integral equation as follows:

$$\frac{\mathbf{M}(r)}{\chi} = \mathbf{H}_s + \frac{1}{4\pi}\nabla\int_{\Omega}\mathbf{M}(r')\cdot\nabla\left(\frac{1}{R}\right)d\Omega \qquad (3)$$

$$\mathbf{H}_s = \frac{1}{4\pi}\int_{\Omega} J_s \times \nabla\left(\frac{1}{R}\right)d\Omega \qquad (4)$$

where the domain of integration is over the iron regions [2]. Equation (4) is the Biot-Savart law for determining the source fields. Equation (3) can also be expressed in terms of the magnetic field vector H.



(a) Geometric model

(b) Comparison with measurements

Fig. 3 Magnetisation Integral Equation Solution[6]

The results shown in Figure 3 are for a magnet designed to detect polarised particles produced in a high energy physics experiment and it can be seen modest agreement was achievable under the pole and in the small beam entry port through the back yoke. Despite the easy modelling and the methods ability to give a good overall assessment of the fields and forces all efforts, at that time, to overcome the basic limitations failed, see Table 1. It was realised that improved accuracy would be obtained if higher order basis functions could be used with Galerkin type weighting rather than the crude point matching used hitherto but this would involve multiple integration's for the matrix coefficients and a more exact treatment of continuity. Notwithstanding the modelling complications the more direct and computer efficient differential method proved in the end to be the way ahead.

Edge Elements

The early success in applying the standard finite element method to differential equations relied on nodally based elements with low order polynomial basis and test functions. One of the more significant recent developments in the numerical solution of field equations has been the introduction of Whitney Forms or 'edge elements' into finite element discretisations, see Bossavit (1988) [3]. The main advantages of this approach are that the correct physical continuity conditions of the field vectors. A, H etc. on material inter- faces are included in the basis functions, and that the local discretised forms, themselves, have divergence and 'curl' properties compatible with the field equations. The edge basis functions have the important physical property that the tangential component of the field is continuous whilst allowing for the possibility of a discontinuity in the normal component. For a full introduction see the technical article by Alain Bossavit in the ICS Newsletter Vol 1, No 3, December 1994.

In the development of edge elements also lay the methodology for improving the quality of integral equation solutions. Already, in hybrid implementations, both methods had been combined by using edge elements in conducting regions and boundary elements in free space [4]. More recently Whitney elements have been applied to the three dimensional integral equation (3) by Kettunen [5]. In this work the earlier limitations have been completely removed with respect to continuity and to point matching and furthermore a full Galerkin projection method was used. Good agreement with measurement and differential codes has been reported for a range of problems (~0.1 %) but, in some cases, there were small oscillations in the solutions near to iron surfaces at the modest discretisations used. The example quoted here is for a 'high field wigler magnet' used to produced synchrotron radiation. The computer model using tetrahedral elements is shown in Figure 4(a) and the field distribution along the line of the beam direction is shown in 4(b); both the total field and the field due to magnetisation is shown (fields from magnetisation of the iron core are directly available from the integral equation whereas a secondary calculation is needed when using the differential approach).

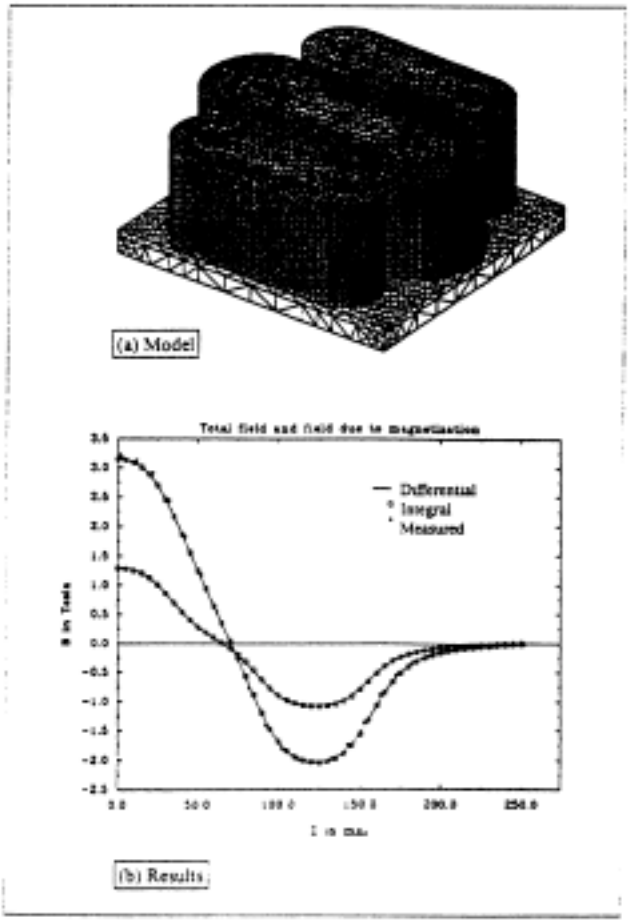


Fig. 4 High Field Wigler central field comparison with integral, differential & measurement

Time dependent problems

The extensions of integral equation methods to two dimensional AC. and transient eddy currents were made in the early 1970's [6]. The three dimensional case proved to be much harder, however for linear non magnetic materials the integral equation circuit analogue methods introduced by Turner (1978) proved practical [7]. The next important development came with the introduction by Albanese & Rubinacci (1988) of using the T-Ω method with edge elements for non-magnetic conducting problems associated with Tokamak design [8]. Their treatment also benefited in part by the earlier pioneering work of Carpenter (1977) on the two component gauge [9] which was used to reduce the number of equations with the tree-co-tree decomposition in order to ensure a unique T for a current density J [8].

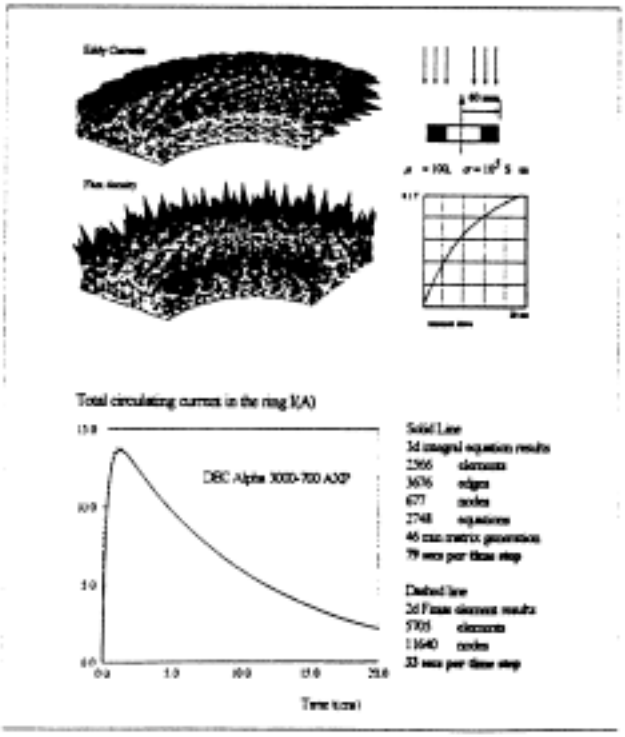


Fig. 5 Conducting Iron ring transient eddy currents

Recently Kettunen & Forsmann (1995) have extended the earlier work on the edge element formulation of the magnetisation integral equation to include the eddy current case[10]. One of their test problems is quoted in

Figure 5 where a solution for a hollow conducting iron ring is compared with a standard finite element solution. This work suggest that equally good solutions can be achieved with either method.

Demanding areas

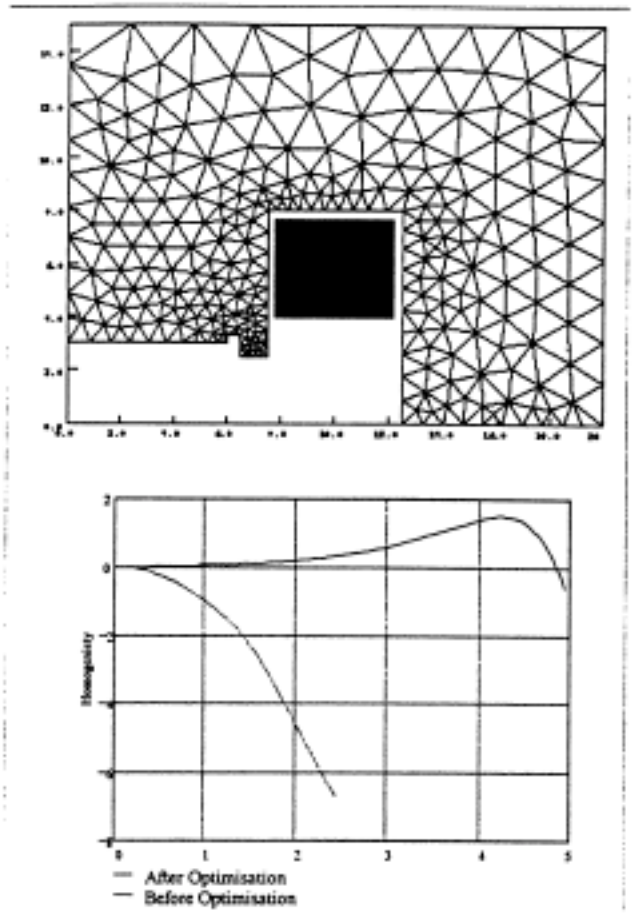


Fig. 6 Field homogeneity (×10⁻⁴) before and after optimisation [11]

There are two very demanding areas in which integral equation methods have a clear modelling advantage. Firstly in problems where material regions are moving with respect to each other the free space mesh needs to be frequently regenerated if a differential method is used. Of course, for uni-directional motion as in a rotating electrical machine or a linear motor this is relatively easy to deal with but the general case is a very different matter. Secondly, in problems of optimisation or synthesis the geometry of regions will change as the optimisation proceeds which also necessitates re-meshing. An example is shown in Figure 6 for a pole-shimming problem where a highly uniform field was required under the pole. A simple integral method was used for equation (3) which included non-linearity so meshing was required throughout the steel but not in the empty space avoiding remeshing except in the shim itself which was straight forward and automatic.

Summary

As the literature shows the use of integral methods for solving electromagnetic field problems has always had many advocates. The method has been widely used in high frequency problems, particularly for radiation [12, 13]. This short article has tried to address some of the arguments for and against their use in practice, particularly for Statics and eddy current problems. For a full treatment on the theoretical background of integral equations in the context of field synthesis see Pawluk [14]. The detailed advantages and drawbacks, over the standard differential approach, are listed in Table 1, but the main points are summarised again, viz.:

a) Only active regions need to be discretised which is a very significant advantage in three dimensions particularly for problems with relative motion, awkward geometry, and in optimisation and synthesis procedures. Furthermore it makes the exchange of data between other engineering applications easier with respect to standards.

b) The far field boundary condition is automatically taken into account by the formulation. In fact this is the same advantage as in (a) but should be highlighted again because accurate solutions of the exterior problem are of great importance in screening, particle beam tracking and in force calculations.

c) The fields recovered from the solution are usually very smooth since the local basis functions are proper solutions of the field equations.

Unfortunately the computational costs are high in all phases to an extent that often only low order basis functions have been used in the past. However since the various algorithms involved are either perfectly parallel, or nearly so, it might be expected that parallel processing would be effective [15]. This is indeed the case and as the new generation of machines, with concurrent processors, become more widely available the expensive variational or Galerkin type integral methods will become more viable. Unfortunately parallel hardware is expensive and as yet it is not cost effective for main-stream industrial users. On the other hand research in linear algebra is yielding several interesting results. For example, iterative solvers based on Krylov methods have recently been tested on non-symmetric linear systems that can arise in solving equation (3) [16]. In this work the authors found that in particular the GMRES (Generalised Minimal Residual) [17] method was orders of magnitude faster than direct methods for large problems. Also the recent research on the application of wavelet transforms as basis functions in finite element discretisation schemes [18] may have an important implication for the speed of solution for integral equations. It appears that the $O(n^3)$ operations needed to solve an integral equation can, in some instances, be reduced to $O(n\log(n))$. These new approaches should make integral methods directly competitive.

As a small gloss on differential forms currently being developed by Hammond and Baldomir, see ICS Newsletter Technical Article, Vol 1, No 2, June 1994 it can be seen that the '1- form H' is the quantity actually being computed in both the edge variable formulations of differential and integral methods reported here. Which may be viewed as a small step toward the rationalisation of electromagnetism?

The aesthetic appeal of integral equation methods is strong not only because of their intrinsic modelling elegance but because the physics involved is directly exploited.

References

[1] G. Strang, *Linear algebra and its applications*, Academic press, New York, 1976.

[2] J. Newman, L.R.Turner, and C. W. Trowbridge, G-FUN: an interactive program as an aid to magnet design, in *Proc. Int. Conf. On Magnet Tech. (MT4)*, pp. 617-626, Brookhaven National Laboratory (1972).

[3] A. Bossavit, Whitney forms: a class of finite elements for three dimensional computations in electromagnetism, *IEE Proc.* Vol 135, pt A, no. 8, pp 493-500, 1988.

[4] A. Bossavit and J.C.Verite, A fixed FEM-BIEM method to solve the 3D eddy currents problem by using **h** as a state variable, *IEEE Trans. Mag.*,Vol.18, 431, 1982.

[5] L. Kettunen, *Volume Integral Formulations for Three Dimensional Electromagnetic Field Computation*, Thesis for Doctor of Technology, Tampere University of Technology, No. 86, 1992.

[6] C.W.Trowbridge, Applications of integral equation methods to magnetostatics and eddy current problems, Chapter 10 in *Finite Elements in Electric and Magnetic Field Problems*, M.V.Chari and P.P. Silvester (Editors), John Wiley, Chichester, 1980.

[7] L Turner, An Integral Equation Approach to Eddy-Current Calculations, *IEEE Trans. Mag.*, Vol 13, p.1119, 1978.

[8] R Albanese and G. Rubinacci, Integral formulation for 3D eddy-current computation using edge elements, *IEE Proc.* Vol. 135, Pt. A, no. 7, pp. 457-462, 1988.

[9] C J Carpenter, Comparison of alternative formulations of 3-dimensional magnetic field and eddy current problems at power frequencies, *IEE Proc.*, vol 124, pp. 1026-1034, November 1977.

[10] L Kettunen & K. Forsman, Integral formulation for 3D eddy current problems, To be published in *IEE Proc.*, 1996.

[11] J Simkin and C W Trowbridge, Optimising electromagnetic devices combing direct search methods with simulated annealing, *IEEE Mag Trans*, Vol 28, March 1992.

[12] Harrington, *Field Computation by Moment Methods*, Macmillan, New York (1968)

[13] P. P. Silvester & R.L.Ferrari, *Finite elements for electrical engineers*, Second Edition, Cambridge University Press, 1990.

[14] K Pawluk, Field Synthesis, In *Computational Electromagnetics*, J K Sykulski (Editor), Chapman & Hall, London, 1995.

[15] C.F. Bryant, M.H. Roberts et al, Implementing a boundary integral method on a transputer system, *IEEE Trans. Mag.*, Vol 26, pp 819-812 (1990).

[16] J Salonen, A Koski et al, Solving Dense Nonsymmet-ric Linear Systems Arising from Integral Equations, *Conference Record, Compumag, Berlin*, pp570-571, 1995.

[17] R Barret, M Berry et al, Templates for the Solution of Linear Systems: Building Blocks for Iterative Methods, *SIAM*, 1994.

[18] G. Beylkin. et al, 1991, Fast Wavelet Transforms and Numerical Algorithms I, *Communications on Pure and Applied Mathematics*, Vol XLIV, p. 141.

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