Three-Dimensional Eddy Current Analysis
By the Boundary Element Method

ABSTRACT

A boundary element method has been developed for the analysis of 3D eddy current problems. This method is based on the boundary integral equation formulation with equivalent electric and magnetic currents and electric charge as unknowns. Linear shape functions, defined parametrically over a quadrilateral element, have been selected for the Galerkin's method.

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Three-Dimensional Eddy Current Analysis By the Boundary Element Method

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Abstract – A boundary element method has been developed for the analysis of 3D eddy current problems. This method is based on the boundary integral equation formulation with equivalent electric and magnetic currents and electric charge as unknowns. Linear shape functions, defined parametrically over a quadrilateral element, have been selected for the Galerkin's method.

I. INTRODUCTION

The numerical treatment of most linear or nonlinear EM field problems can be effectively achieved through a boundary element method (BEM) [1]. The boundary element method, as compared with domain-type methods, such as finite element method (FEM) and finite difference method (FDM), has two salient advantages: 1) dimensions of the problem are effectively reduced by one; 2) the analysis is equally applicable to bounded and unbounded regions. In the case of analysis of eddy current problems with high conductivity or at high frequency, domain type methods are very difficult and expensive to use in handling strong skin effect and open boundary problems.

There are three major difficulties involved in the analysis of 3D eddy current problems using BEM: 1) the vector nature of problems results in a very large number of unknowns; 2) the loose coupling between electric and magnetic fields in the air at low frequency makes it difficult to satisfy the interface conditions and to obtain a unique solution in conjunction with an appropriate choice of gauge condition; 3) highly singular kernels in boundary integral equations require expensive numerical integration to obtain accurate solutions.

In recent years, different formulations for 3D eddy current problems have been developed. Mayergoyz [2] has formulated an integral equation in terms of the equivalent electric current and magnetic charge. The greatest advantage of this formulation is that only three unknowns are required on the node on an interface between a nonconducting region and a conducting region. However, in general this formulation can only handle problems with simply connected regions. Moreover, the equations involve highly singular terms that must be integrated in the Cauchy's principal sense. This makes if very difficult to use the Galerkin's method to solve the equations. Another minimum order formulation has been developed by Shao, Zhou and Lavers [3], using the second order vector potential in the conductor region and a magnetic scalar potential in the air. This formulation also contains highly singular integrals. Morisue [4] has presented a formulation using the magnetic vector potential and the electric scalar potential. A set of boundary conditions are used to satisfy the Coulomb gauge in order to give a unique solution for the problem with multiply connected regions. A total of 12 unknowns are required on each node and highly singular integrals are involved in the formulation. Rucker and Richter [5] have developed a formulation using the magnetic flux, magnetic vector potential and electric scalar potential as unknowns. This method requires 7 unknowns on each node and also involves highly singular integrands. Ishibashi has used a least residual approach in conjunction with BEM to analyze 3D eddy current problems [6]. However, this formulation is quite expensive because of the involvement of the least squares solutions.

In this paper, a boundary element method is presented for the analysis of time harmonic 3D eddy current problems. The method is based on the boundary integral equation formulation with the equivalent electric and magnetic currents, and electric charge as unknowns. This integral equation formulation is preferable to the several other possible variants because it can handle multiply connected regions and has only R^{-1} singular kernels in the integrals. In the BEM technique, the surface of a conducting object is modeled in terms of curvilinear quadrilateral elements. The Lagrange linear shape functions, defined parametrically over each element, have been selected for the Galerkin's method. A numerical example is given to show the accuracy and reliability of this method.

II. INTEGRAL EQUATION FORMULATION

Consider a linear, isotropic and homogeneous conducting material region V_c embedded in free space V_o. The region V_c is bounded by the surface S with the outward unit normal $\hat{n}$, and is characterized by permeability μ and σ conductivity . Assume there exists an external excitation (B^1, E^1) with $e^{j\omega t}$ variation. The displacement current is neglected since we are only interested in eddy current problems at low frequency in this paper.

Using the vector Green's theorem, the ***B*** and ***E*** fields at an arbitrary observation point ***r*** in V_o can be expressed as [6]

$$cB_1(r) = B^i(r) + \int_S \hat{n}' x B_1(r') x \nabla' G_1 ds' + \int_S \hat{n}' \cdot B_1(r') \nabla' G_1 ds' \tag{1}$$

$$cE_1(r) = E^i(r) - j\omega \int_S \hat{n}' x B_1(r') G_1 ds' + \int_S \hat{n}' x E_1(r') x \nabla' G_1 ds' + \int_S \hat{n}' \cdot E_1(r') \nabla' G_1 ds' \tag{2}$$

where c=1 for r in V_0 and c=0.5 for r on the surface S. Note that when r is on the surface S, the surface ingegrals in (1) and (2) are the principal value integrals which exclude the contribution from the singularity point.

Similarly, the fields inside the conductor V_c can be written as

$$cB_2(r) = \int_S \hat{n}' x B_2(r') x \nabla' G_2 ds' - \int_S \hat{n}' \cdot B_2(r') \nabla' G_2 ds' - \mu\sigma \int_S \hat{n}' x E_2(r') G_2 ds' \tag{3}$$

$$cE_2(r) = j\omega \int_S \hat{n}' x B_2(r') G_2 ds' - \int_S \hat{n}' x E_2(r') x \nabla' G_2 ds' - \int_S \hat{n}' \cdot E_2(r') \nabla' G_2 ds' \tag{4}$$

In the above equations G_1 and G_2 are the Green's functions in region V_0 and V_c, respectively, and are given as

$$G_1 = \frac{1}{4\pi R} \qquad and \qquad G_2 = \frac{e^{-jkR}}{4\pi R} \tag{5}$$

where R is the distance between the observation point r and source (integrating) point r', and $\kappa^2 = -j\omega\mu\sigma$. It is important to notice that using (1)-(4), we have [6]

$$B_1(r) = E_1(r) = 0, \qquad r \in V_c \tag{6}$$

$$B_2(r) = E_2(r) = 0, \qquad r \in V_o \tag{7}$$

Now we can define the equivalent electric and magnetic surface currents as

$$J_s(r) = \hat{n} x H_1(r) \tag{8}$$

$$M_s(r) = -\hat{n} x E_1(r). \tag{9}$$

Since the tangential components of E and H are continuous across S, J_s and M_s can also be expressed as

$$J_s(r) = \hat{n} x H_2(r) \tag{10}$$

$$M_s(r) = -\hat{n} x E_2(r). \tag{11}$$

The normal component of the B field can be expressed as

$$\hat{n} \cdot B_1 = \hat{n} \cdot B_2 = \frac{1}{j\omega} \nabla \cdot (\hat{n} x E_1) = \frac{j}{\omega} \nabla \cdot M_s. \tag{12}$$

Generally speaking, one can obtain the normal component of E from the tangential components of H using

$$j\omega\varepsilon_0 \hat{n} \cdot E_1 = -\nabla \cdot (\hat{n} x H_1) = -\nabla \cdot J_s. \tag{13}$$

However, for low frequency eddy current problems, eq. (13) is not adequate to obtain the ***E*** normal due to the nature of $\omega\varepsilon_0 \approx 0.$ Therefore, we need to define the equivalent electric charge as

$$q_e = \hat{n} \cdot E_1(r) \varepsilon_0. \tag{14}$$

Also, $\hat{n} \cdot E_2 = 0$ should be used in (4).

At this point, one may want to use two of the equations in (1)-(4) to solve the equivalent sources. However, as discussed in [6], in order to obtain an accurate solution, an over determined system has to be solved in the least squares sense since there are more equations that have to be satisfied than unknowns. Furthermore, since this approach involves highly singular kernels in the integral equations, it becomes difficult to implement using the Galerkin's method. In this paper, we use a different approach to obtain the boundary integral equations.

Substituting the equivalent sources in (1)-(4), and letting the r approach the surface from the positive side of S in (1) and (2), and from the negative side in (3) and (4), and adding (1) to (3) and (2) to (4), with some manipulations we obtain a set of boundary integral equations as

$$\hat{n} x \left\{ \int_S (\mu_0 \nabla G_1 - \mu \nabla G_2) x J_s ds' + \int_S \mu\sigma G_2 M_s ds' + \frac{1}{j\omega} \int_S (\nabla G_1 - \nabla G_2) \nabla'_s \cdot M_s ds' \right\} - (\mu_0 + \mu) \frac{J_s}{2} = -\hat{n} x B^i \tag{15}$$

$$\hat{n} x \left\{ -j\omega \int_S (\mu_0 G_1 - \mu G_2) J_s ds' - \int_S (\nabla G_1 - \nabla G_2) x M_s ds' - \int_S \nabla G_1 \frac{q_e}{\varepsilon_0} ds' \right\} + M_s = -\hat{n} x E^i \tag{16}$$

$$\hat{n} \cdot \Big\{ - j\omega \int_s (\mu_0 G_1 - \mu G_2) J_s ds' - \int_s (\nabla G_1 - \nabla G_2)$$

$$x M_s ds' - \int_s \nabla G_1 \frac{q_e}{\varepsilon_0} ds' \Big\} - \frac{q_e}{2\varepsilon_0} = -\hat{n} \cdot E^i \tag{17}$$

To use (15)-(17) in a boundary element method, one has to use the shape function so that $\nabla_s \cdot M_s$ exists and is continuous on the surface S. If $\nabla_s \cdot M_s$ is discontinuous along some lines, additional line integrals are required in (15). Instead of using (15), one can transfer the operator nabla on the magnetic current to act on the Green's functions. Thus, eq. (15) can be expressed as

$$\hat{n} x \Big\{ \int_s (\mu_0 \nabla G_1 - \mu \nabla G_2) x J_s ds' + \int_s \mu\sigma G_2 M_s ds' + \frac{1}{j\omega}$$

$$\int_s (M_s \cdot \nabla)(\nabla G_1 - \nabla G_2) ds' \Big\} - (\mu_0 + \mu) \frac{J_s}{2} = -\hat{n} x B^i \tag{18}$$

A similar approach has been used by Muller [8] for the analysis of high frequency scattering problems where (17) is not required since one can obtain the E normal from the tangential components of the H field using (13).

One can prove that, except for the third term on the left hand side of (16), the highest singularity in all of the kernels in (15)-(18) is R^{-1}, which can easily be handled using standard numerical integration techniques. Since the Galerkin's method has been used in our boundary element method, double surface integrations are involved in the matrix accumulation. Applying Stokes theorems to the third term of (16), one can transform one of the surface integrals into a line integral, which effectively reduces the singularity to R^{-1}.

The above formulation can be applied to general cases. For example, we can use (15) or (18) to the surface of a non-conducting magnetic material region where only J_s exists. One can easily prove that (15), in this case, can be written as

$$\hat{n} x \int_s (\mu_0 - \mu) \nabla G_1 x J_s ds' - (\mu_0 + \mu) \frac{J_s}{2} = -\hat{n} x B^i \tag{19}$$

which is the same as the equation used in magneto static problems [1]. If there exists an interface between two conductors with different conductivities, one needs to apply (15) or (18) and (16) to the surface with a minor modification. In this case, the normal component of the E field can be obtained using

$$\hat{n} \cdot E_1 = -\frac{\nabla_s \cdot J_s}{\sigma_1} \text{ and } \sigma_1 \hat{n} \cdot E_1 = \sigma_2 \hat{n} \cdot E_2 \tag{20}$$

III. Implementation and Results

The boundary integral equations in (16)-(18) are solved using a boundary element method. The surface S is modeled using the isoparametric, quadrilateral elements [1]. Linear vector shape functions are used to represent the J_s and M_s, and a linear scalar shape function is used to model the electric charge q_e. Five unknowns are required on each node. If there are N nodes in a problem, a system of equations with 5Nx5N coefficients can be obtained using the Galerkin's method. Then this system of equations can be effectively solved either directly or iteratively. The fields of interest at any point can be easily calculated using (1), (2), (3), or (4).

As a numerical example, we use the present method to solve the TEAM problem 3 [9]. This problem consists of a conducting bath plate with 2 holes excited by a circular coil as show in Figs. 1 and 2. The conductivity of the plate is $\sigma = 3.27x10^7$ S/m and the coil carries a current of I=1260 A turns. The analysis has been done for two coil positions and at two frequencies. Making use of the symmetry of the problem for coil position 1, only a quarter of the late needs to be analyzed, and 152 elements are used. In the case of coil position 2, one half of the plate has been modeled, and 304 elements are used.

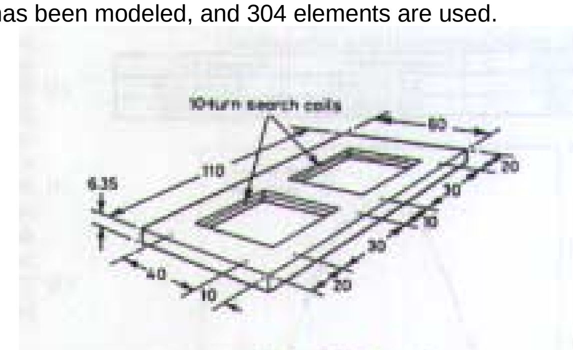

Fig. 1 Conducting plate with two holes (1=mm).

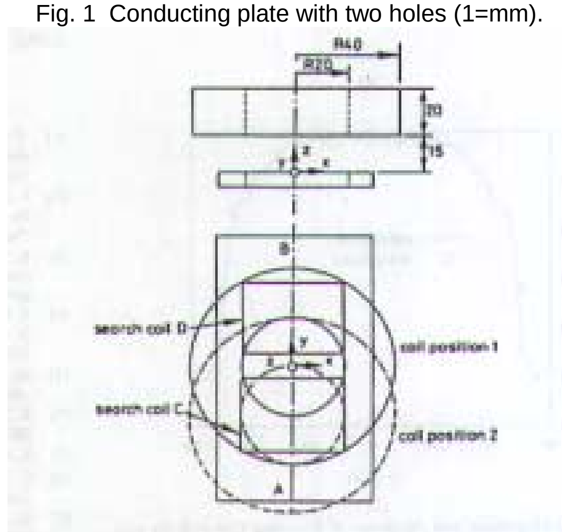

Fig. 2. Conducting plate and the two excitation coil positions.

We list the calculated and measured [10] induced voltages in 10-turn search coils in Table I. In Fig. 3, we show calculated and measured flux densities, Bz, along the line A-B (x=0, z=0.5 mm) for coil position 1 at f=50 Hz.

Table I (a): Voltages in search coil C due to coil at position 1

Frequency (Hz)	Calculated (mV)	Measured (mV)
50	21.61 <-108.4>	22.10 <-109.1>
200	54.24 <-141.7>	54.32 <-142.0>

Table I (b): Voltages in search coil C due to coil at position 2

Frequency (Hz)	Calculated (mV)	Measured (mV)
50	30.13 <-107.5>	30.74 <-108.0>
200	78.15 <-140.9>	78.43 <-142.2>

Table I (c): Voltages in search coil D due to coil at position 2

Frequency (Hz)	Calculated (mV)	Measured (mV)
50	5.5 <-119.6>	6.3 <-118.4>
200	10.3 <-173.7>	11.17 <-166>

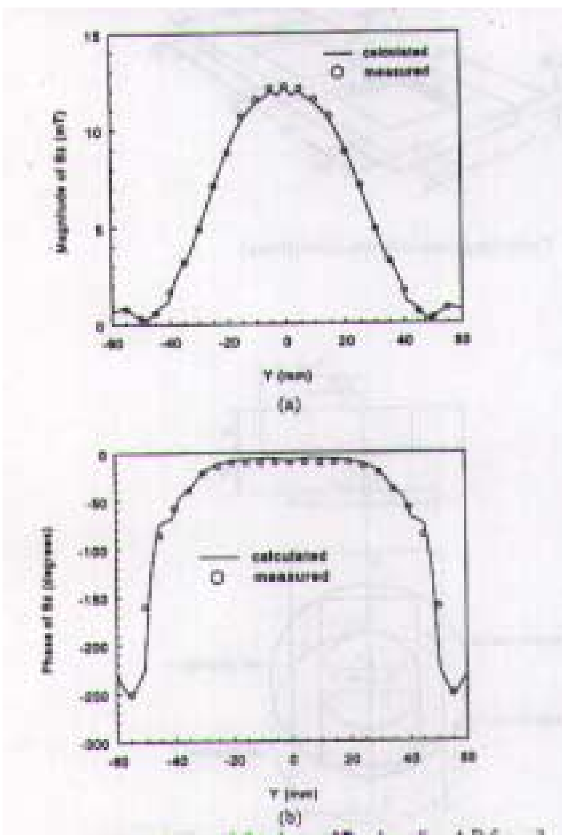

Fig 3. (a) Magnitude and (b) phase of Bz along line A-B for coil position 1 at f=50 Hz.

Good comparison between calculated and measured results has been observed in general.

IV. Conclusion

A general and accurate boundary element method is presented for the analysis of 3D eddy current problems. The advantages of this newly developed method are as follows:

(1) The correct fields E and B can easily be calculated at any point after the equivalent sources are obtained;
(2) The method is general and can be used to solve problems with multiply connected regions and other complicated configurations;
(3) Since R^{-1} singularity kernels are involved in the integral equations, efficient and simple numerical integration algorithms can be employed to obtain very accurate results;
(4) Since this formulation uses the Fredholm integral equation of the second kind, the resulting matrix is diagonally strong and can be effectively solved iteratively in most cases;
(5) The use of the Galerkin's method leads to faster solution convergence and more reliable results.

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