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COMPUTER-AIDED FIELD ANALYSIS OF HIGH VOLTAGE APPARATUS USING THE BOUNDARY ELEMENT METHOD

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ABSTRACT

The boundary element method (BEM) and its use in computer-aided field analysis is presented in this paper. The BEM is compared against the finite difference method (FDM) and finite element method (FEM) for two-dimensional and rotationally symmetric problems. The advantages of BEM are stated for an application in high voltage power apparatus design. It is shown that BEM is superior to FDM and FEM, both for linear and non-linear problems.

INTRODUCTION

The common concept in the numerical methods is the reduction of the governing field equation or an equivalent integral formulation into a linear system of equations. These methods can be classified in two categories: the methods where approximations are to be made throughout the region B, and the methods where approximations are to be made only on the boundary ∂B. The finite difference and finite element methods belong to the first category while the boundary element methods belong to the second.

BOUNDARY ELEMENT METHOD

The methods of the second category solve a boundary integral equation formulation of the problem for some unknowns on $\partial\beta$ ([1]. These methods not only produce precise results with far less data as compared to the methods of finite differences and finite elements but also cater to open region problems without any artificial truncation of the region and model problem geometries accurately. Since the approximations are done only on the boundary, the dimensionality of the problem is reduced by one. Furthermore, usually being bounded and often completely continuous, integral operators as compared to differential operators admit a wider selection of trial functions [2].
Direct methods in this category solve an integral equation formulation for the unknowns directly [3], while indirect methods solve for the source of the unknown [4]. The boundary element method presented in this paper is an indirect method. An equivalent source, which would sustain the field, is found by forcing it to satisfy prescribed conditions under a free space Green's function which relates the location and effect of the source to any point on the boundary.
The use of Green's function effectively eliminates the need for a finite element mesh or a finite difference grid.

Once the source is determined, potential and field are computed by integrating the source without interpolation. This provides inherent stability. Capacitance, inductance, and other parameters can be calculated by integrating the free charge, which is derived from the equivalent source [5]. Provided the problem is piecewise homogeneous, the equivalent source is located only on the boundaries and interfaces of different media.

In noon-linear problems, the BEM sill solves for the source of the field and not for its potential. All advantages of calculating the source applies. Only regions with non-linearities contain volume unknowns [6].

PHYSICAL BASIS

In the electrostatic field,

$$\Delta X E=0, \qquad (1)$$

so that E is irrotational and hence conservative which is a necessary and sufficient condition for the existence of a potential Φ in the form

$$E= -\Delta\Phi \qquad (2)$$

According to maxwell's equation, in a source free region

$$\Delta . D=0. \qquad (3)$$

The constitutive relation for a linear, isotropic region of dielectric constant ε is

$$D=\varepsilon E \qquad (4)$$

If the region is homogeneous, combining (2), (3) and (4)

$$\Delta^2\Phi=0 \qquad (5)$$

INTEGRAL EQUATION FORMULATION

In a bounded region B with a piecewise smooth boundary ∂B, application of Green's theorem [12]

$$\iint_B (\Phi\,\Delta^2 G - G\Delta^2\Phi)\partial B = \int_{\partial B}(\Phi\frac{\partial G}{\partial n} - G\frac{\partial \Phi}{\partial n})dr' \qquad (6)$$

to the unknown potential Φ and the free space Green's function [13]

$$G\,(r, r')= \ln \frac{k}{|r-r'|} \qquad (7)$$

satisfying

$$-\Delta^2 G=2\prod\delta \qquad (8)$$

Where δ is the Dirac Delta function, yields

$$\int_{\partial B}(\Phi \frac{\partial G}{\partial n} - G\frac{\partial \Phi}{\partial n})\, dr' = -2\pi\Phi \text{ when } r\varepsilon\text{B} \qquad (9a)$$

$$\int_{\partial B}(\Phi \frac{\partial G}{\partial n} - G\frac{\partial \Phi}{\partial n})\, dr' = 0 \text{ when } r\varepsilon B_0 \qquad (9b)$$

k is a constant chosen such that k> max | r-r' | which ensures that Green's function is strictly positive throughout B.

B_0 is the region exterior to B. The validity of (9 a and b) can be extended to an infinite region provided that Φ and G are regular at infinity [7]. Thus for exterior region

$$-\int_{\partial B}(\Phi_0 \frac{\partial G}{\partial n} - G\frac{\partial \Phi_0}{\partial n})\, dr' = 0 \qquad \text{when } r\varepsilon\text{B} \quad \text{and} \quad (10a)$$

$$-\int_{\partial B}(\Phi_0 \frac{\partial G}{\partial n} - G\frac{\partial \Phi_0}{\partial n})\, dr' = -2\pi\Phi \qquad \text{when } r\varepsilon B_0 \qquad (10b)$$

From equations (9) and (10)

$$-\int_{\partial B}(\Phi - \Phi_0\ )\frac{\partial G}{\partial n} - G(\frac{\partial \Phi}{\partial n} - \frac{\partial \Phi_0}{\partial n})\, dr' = \Phi(r) \qquad (11)$$

The choice of $\Phi=\Phi_0$ and

$$\sigma\ (r')= (\frac{\partial \Phi}{\partial n} - \frac{\partial \Phi_0}{\partial n}) \qquad (12)$$

$$\text{gives} \int G\ (r, r')\ \sigma\ (r')dr' = \Phi\ (r) \qquad (13)$$

which is a single layer integral equation formulation for the Laplace's equation [1] and [4]

The integrand contains the distributed source and the free space Green's function. From (13), given the source configuration, the potential can be found everywhere in the region. Usually, however, the source is not known but the potential or its normal derivative are specified on the boundary, and we seek an equivalent source that will sustain these conditions. Once, the equivalent source is known, any field value or parameter can be calculated.

For Dirichlet boundaries the equation to be enforced is (13). For exterior Dirichlet problems to construct an acceptable solution the boundary decomposition (1)

$$\Phi(r)= \int G\ (r, r')\ \sigma\ (r')dr' + c \quad r\ \varepsilon\ \partial B \qquad (14)$$

is introduced where c is a constant to be determined under the side condition,

$$\int \sigma\ (r')\ dr' = 0 \qquad (15)$$

(15) is necessary for the logarithmic potential to be regular at infinity.

For Neumann boundaries a Fredholm equation of the second kind results

$$\Phi'(r)= \int G\ '(r, r')\ \sigma\ (r')dr' + \pi\sigma(r) \qquad r\varepsilon\partial B \qquad (16)$$

Where Φ'(r) and G' (r,r') are the normal derivative of the potential and Green function with respect to the unprimed variable.

Along any interface the continuity of flux density is enforced yielding

$$(\varepsilon 1-\varepsilon 2)\int G'^{(r,r')}dr'\sigma(r')dr' + (\varepsilon 1+\varepsilon 2)\sigma(r) = 0$$

Where $\varepsilon 1$and $\varepsilon 2$ are the permittivity values of the materials forming the interface. To solve above integral equations for the equivalent source the Galerkin method is used.

PROJECTION METHODS

Projection methods are also called method of weighted residuals or moment methods (3), (4), (9). Consider the operator equation

$$L\sigma=g \qquad (18)$$

Where L is assumed to be a linear operator which maps XX to g uniquely. Normally L and g are known and we have the deterministic problem of finding σ. That is, we are required to solve

$$\sigma=L^{-1}g \qquad (19)$$

Where L^{-1} is assumed to exist and that the solution for σ is unique.

Let the solution be expanded by the series of functions in the domain of the operator and let a_1, a_2 and a_3 ……. be coefficient such that

$$\sigma(x)= \sum_{n=1}^{m} a_n b_n(x) \qquad (20)$$

For an exact solution the expansion functions must form a complete set which is usually infinite in number. Rewriting (18) as

$$L\sigma(x)-g(x)=0 \qquad (21)$$

And substituting the expansion functions to approximate the potential, the residual is

$$R= \sum_{n=1}^{m} a_n b_n(y)-g(x) \qquad (22)$$

which is equal to zero only if the coefficients and expansion functions can be found such that they are the exact solution. In the projection method the coefficients are found in such a way that the residual is forced to be zero – giving the best approximation.

A suitable inner product is taken with the residual and some prescribed functions over the range of the operator. These functions are called weighting functions, or more descriptively, testing functions. The inner product is defined by

$$<w_m,R>=\int R\, w_n \, ds \qquad m=1,2,3\ldots\ldots \qquad (23)$$

where $w_1,w_2,w_3\ldots$.are are the testing functions. The inner product is set to zero forcing the residual to be orthogonal to the testing functions

$$<w_m,R>=0 \qquad (24)$$

Substituting (22) into (24) and rearranging yields

$$\sum_{n=1}^{m} a_n < w_m\,, Lbn(x) > = < w_m, g(x)> \qquad (25)$$

For a solution of (25) we approximate (20) by a finite sum. Eq. (25) is then a finite set of linear equations which can be put in matrix form as

$$S_a=b \qquad (26)$$

where

$$S_{mn}=<w_m,Lb_n>$$
$$b_m=<w_m,g) \qquad (27)$$

Assuming the matrix is not singular it may be inverted yielding the coefficients. These coefficients may then be substituted into (20) giving an approximate (on rare occasion on exact) solution for the charge.

The accuracy of the approximation will obviously depend upon the choice of the expansion and testing functions, and the number of them used. These coordinate functions must be linearly independent as linear dependence will result in a singular S matrix.
The particular choice of the expansion functions being the same as the testing functions is called Galerkin's method.

Boundaries are discretized into individual sections which are referred to as boundary elements. The expansion and testing functions, as well as the geometry, are specified on an element-by-element basis. Coefficients of the expansion functions are normally defined at nodes on the element. Each node is associated with a particular expansion function. Using linear shape functions,

$$\alpha_1=1-\xi$$
$$\alpha_2=\xi \qquad (28)$$

The charge over each element is expressed as

$$\sigma=\sum_{i=1} \sigma_1 \alpha_1(\xi)$$

where m=2 as linear elements are used.

Using Lagrange quadratic shape functions

$$\alpha_1=2\xi^2-3\xi+1$$
$$\alpha_2=4(\xi-\xi^2) \qquad (30)$$
$$\alpha_3=2\xi^2-\xi$$

over the domain (0, 1), global positions in cartesian coordinates are specified parametrically over each element as

$$x=\sum_{i=1}^{m} \alpha_1 (\xi) x_1$$
$$y=\sum_{i=1}^{m} \alpha_1 (\xi) y_1$$

we wish to determine

$$<w_m,L\sigma_n>=<w_m,b>$$

Which can be put in vector notation as

$$<\alpha,L\alpha^T>\sigma=<\alpha,g>$$

The operator L is dependent upon the boundary conditions where the inner product is being calculated.

MICROCOMPUTER IMPLEMENTATION

With the advent of powerful microcomputers, computations that were once only possible on mini and mainframe computers, are now possible on microcomputers. In addition, microcomputers offer highly interactive graphics capabilities which can be an invaluable aid in the design of a system.

The boundary element method presented above has been implemented, on a microcomputer, in the program ELECTRO. The geometry of the problem that can be solved is arbitrary. The conductors may be of finite area or infinitesimally thin.

The solver steps over each element and applies the appropriate inner product. All the integrals are calculated over the simplex (0, 1).

One difficulty is the integration of the Green's function singularity which occurs when the observation and source points coincide. This problem is easily catered to by dividing out the singularity and using a quadrature scheme containing the form of the singularity. This technique enables very accurate integrations of the singular integrand.

Representing the potential at each point by a phasor, steady state sinusoidal fields can be calculated with ease. For example, multiphase transmission line fields can be calculated by solving a set of real and imaginary equivalent sources for given complex boundary conditions.

The special features of the microcomputer environment, e.g., fast color graphics, color printer, mouse or keyboard entry, math coprocessor, hard disk, RAM disk, are fully utilized to create an integrated package which includes problem definition, analysis, data storage and transfer, drafting and presentation capabilities.

The user interface has been designed to require minimal keyboard entry and hand motion. Menus are structured to follow the natural pattern of defining and solving a problem and to incorporate the same sets of commands that operate on different objects. On-line help is provided in every menu. The use of the boundary element method also benefits the user interface. Geometry definition is not built around a mesh and the accuracy of results is easily checked by sound means.

One has very powerful options to test the accuracy of a solution. On boundaries, the calculated and assigned conditions can be compared. Along interfaces, the calculated and actual field discontinuities can be checked. One could test the field values inside conductors. According to maximum principle of harmonic functions, the largest errors occur on boundaries. Hence these checks indicate the largest error in a solution.

In the FEM, since the results are provided by interpolation no such simple and quick ways of checking the accuracy of a solution exits – on boundaries one would obtain exactly what was assigned.

APPLICATION

A Bus Bar Problem

Maximum value of electric field magnitude (kV/mm) is calculated for a pair of rectangular bus bars shown in Fig. 1.

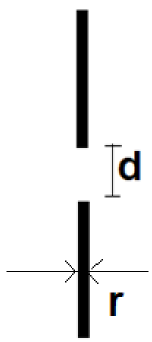

Fig. 1: Rectangular Bus bar configuration.

Table 1 presents the results for varying distance d and corner radius r. Dimensions are in inches.

Table 1

Bus bar problem results.

d	r	ELECTRO	FEM
2.0	0.125	2.6	2.7
2.0	0.0625	2.9	3.3
2.0	0.0	9.6	5.9
2.5	0.0	8.6	5.0
2.8	0.0	8.2	4.5
3.0	0.0	7.9	4.3
3.5	0.0	7.3	3.8

When r = 0, BEM consistently gives higher values as the number of elements is increased. This is expected as the number of elements is increased. This is expected as the field is infinite at the corner. When corners are rounded, FEM gives values higher than BEM, possibly due to the artificial truncation of the open region.

CONCLUSIONS

The boundary element has been shown to be an efficient technique for the solution of Laplace's equation for piecewise homogeneous media. This is mainly due to the reduction of one in dimensionality as all the unknowns are located only on the boundaries and interfaces. This differs from the finite difference and finite element methods in which the whole must be discretized. The unknown, computed using the boundary element method, is the equivalent charge that sustains the field. Once the equivalent charge is known any parameter can be derived.

(1) M.A. Jaswon and G.T. Symm, Integral Equation Methods In Potential Theory and Elastostatics. New York: Academic Press, 1977.
(2) S.G. Mikhlin, Integral Equations. New York: Pergamon Press, 1964.
(3) Y.B. Yildir, A Boundary Element Method for the Solution of Laplace's Equation in Three-Dimensional Space. Ph.D. Dissertation, University of Manitoba, 1985.
(4) B.W. Klimpke, A Two-dimensional Multi-media Boundary Element Method. M.Sc. Dissertation, University of Manitoba, Winnipeg, Man., Canada R3T 2N2, 1983.
(5) C. Wei, R.F. Harrington, J.R. Mautz and T.K. Sarkar, "Multiconductor Transmission Lines in Multilayered Dielectric Media," IEEE Trans. Microwave Theory Tech., Vol. MIT-32, pp.439-449, April, 1984.
(6) M.H. Lean and D.S. Bloomberg, "Nonlinear Boundary Element Method for Two-dimensional Magnetostatics," J. Appl. Phys., Vol. 55, No.6, 1984.
(7) J.A. Stratton, Electromagnetic Theory. New York: McGraw-Hill Book Co., 1941.
(8) R.F. Harrington et al. "Computation of Laplacian Potentials by an Equivalent Source Method," Proc. IEE, Vol. 116, No. 10, 1715-1720, 1969.
(9) R.F. Harrington, Field Computation by Moment Methods. Reprinted by R.F. Harrington, R.D.2, West Lake Road, Cazenovia, N.Y., 1968.